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A note on a proper holomorphic mapping from an n-ball to a k-ball $(k>n)$.

Dedicated to Dr. Eiichi Sakai for his 70th. birthday.

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1. It is well known that any proper holorphic mapping from a unit n-ball to a unit n-ball is biholomorphic if $n\geqq 2$. For the non-equidimensional case, this is not true ([1]). But any proper holomorphic mapping from a unit n-ball to a unit $(n+1)$-ball which is of class C^2 on its boundary is essentially linear isometry ([1], Theorem 4) if $n\geqq 3$. In this note, using the same idea of [1], we shall show that any proper holomorphic mapping from a unit n-ball to a unit k-ball $(k>n)$ which is of class C^2 on its boundary is linear isometry if it has some linear algebraic conditions.

2. Let B_n be a unit ball of $\mathbf{C}^n$, that is,

$$\mathrm{B}_n=\{z\in\mathbf{C}^n;\ |z_1|^2+\cdots+|z_n|^2<1\}.$$

For a point a of B_n, an involution ϕ_a is defined by

$$\phi_a(z)=\frac{a-\mathrm{P}_a(z)-s\mathrm{Q}_a(z)}{1-<z,\ a>},\ \text{where}$$

$$\mathrm{P}_a(z)=\frac{<z,a>}{|a|^2}a,\ s=\sqrt{1-|a|^2},\ \mathrm{Q}_a(z)=z-\mathrm{P}_a(z)$$

and $<z,\ a>$ is an hermitian inner product, $|z|^2=<z,\ z>$.

It is easily seen that $\phi_a\cdot\phi_a$ is an identity mapping and $\phi_a(a)=0$, the origin of $\mathbf{C}^n$.

Remark 1. For points a, z of B_n, put $\tilde{a}=(a,\ o')\in\mathrm{B}_k$, $\tilde{z}=(z,\ o')\in\mathrm{B}_k$. Then it holds that $\phi_{\tilde{a}}(\tilde{z})=(\phi_a(z),\ o')$.

Let $\mathbf{U}$ be a domain of $\mathbf{C}^n$, f be a holomorphic mapping from $\mathbf{U}$ into $\mathbf{B}^m$.

Definition 2. $\mathrm{D}^kf(u)(v^k)=\sum_{|\mathrm{I}|=k}\frac{\partial^kf(u)}{\partial z_1^{i_1}\partial z_2^{i_2}\cdots\partial z_n^{i_n}}v_1^{i_1}v_2^{i_2}\cdots v_n^{i_n}$ where $u\in\mathbf{U}$, $v=(v_1,\ v_2,\ \cdots,\ v_n)$ and $|\mathrm{I}|=i_1+i_2+\cdots+i_n$.

Definition 3.
$$\mathrm{D}^kf(u)(\tau^{k-\ell})=\frac{1}{\binom{k}{\ell}}\sum_{|\mathrm{I}|=k}\left(\frac{\partial^kf(u)}{\partial z_1^{i_1}\partial z_2^{i_2}\cdots\partial z_n^{i_n}}\times\sum_{|S|=\ell}\binom{i_1}{s_1}\cdots\binom{i_n}{s_n}\tau_1^{i_1-s_1}\eta_1^{s_1}\cdots\tau_n^{i_n-s_n}\eta_n^{s_n}\right)$$

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where $\binom{k}{\ell}=\frac{k!}{(k-\ell)!\ell!}$ and $\tau=(\tau_1, \tau_2, \cdots, \tau_n)$, $\eta=(\eta_1, \eta_2, \cdots, \eta_n)$.

Remark 4. By an easy calculation, we have

$$D^k f(u)((\tau+\eta)^k)=\sum_{\ell=1}^{k}\binom{k}{\ell}D^k f(u)(\tau^{k-\ell}, \eta^\ell).$$

Remark 5. The Frechet derivatives of an involution ϕ_a sftisfy

$$D^k\phi_a(z)(v^k)=k!\left[\frac{\langle v, a\rangle}{1-\langle v, a\rangle}\right]^{k-1}D\phi_a(z)(v).$$

Let f be a proper holomorphic mapping from a unit ball B_n to a unit ball B_k $(k>n)$ which is of class C^2 on $\bar{B}_n$. We may assume $f(o)=o$. We fix an orthonormal basis $\{u, \tau^{(2)}, \cdots, \tau^{(n)}\}$ of C^n. Now suppose that there exists a vector $v\in C^n-\{o\}$ such that for any choice of an automorphism ϕ of B_n and a ψ of B_k with $\psi\circ f\circ\phi(o)=o$, the following condition (*) is satisfied:

(*) $D^2(\psi\circ f\circ\phi)(u)(v^2)$ is in the space spanned by $\{\psi\circ f\circ\phi(u), D(\psi\circ f\circ\phi)(u)(\tau^{(2)}), \cdots, D(\psi\circ f\circ\phi)(u)(\tau^{(n)})\}$.

Then it holds that f is linear and isometry, that is $|f(z)|=|z|$. We shall prove this. Since the rank of the linear mapping $Df(u)$ is equal to n ([1], page 492), we can choose a point $a\in B_n$ such that the rank of $Df(a)$ is equal to n. Take involutions ϕ of B_n and a ψ of B_k such that $\phi(o)=a$, $\psi(f(a))=o$ and put $g=\psi\circ f\circ\phi$. Then $D^2g(u)(v^2)$ is in the space spanned by $\{g(u), Dg(u)(\tau^{(2)}), \cdots, Dg(u)(\tau^{(n)})\}$. Since n vectors $\{g(u), Dg(u)(\tau^{(2)}), \cdots, Dg(u)(\tau^{(n)})\}$ are lineary independent, the rank of the $(n+1, k)$ matrix

$$\begin{pmatrix} g(u) \\ Dg(u)(\tau^{(i)})\ 2\leqq i\leqq n \\ D^2g(u)(v^2) \end{pmatrix}$$

is equal to n. Let $1\leqq i_1\leqq i_2\cdots<i_{n+1}\leqq k$ and put $G=(g_{i_1}, \cdots, g_{i_{n+1}})$.
Then

$$\det\begin{pmatrix} G(u) \\ DG(u)(\tau^{(i)})\ 2\leqq i\leqq n \\ D^2G(u)(v^2) \end{pmatrix}=0.$$

Fix a complex number λ with $|\lambda|=1$. Since $g(\lambda z)$ satisfies the condition (*), by an easy calculation,

$$\det\begin{pmatrix} G(\lambda u) \\ DG(\lambda u)(\tau^{(i)})\ 2\leqq i\leqq n \\ D^2G(\lambda u)(v^2) \end{pmatrix}=0.$$

This holds for all $\lambda\in C$ with $|\lambda|=1$ and since the left handside of the above is holomorphic in $|\lambda|<1$, the above equality holds for all $\lambda\in C$ with $|\lambda|\leqq1$. Therefore we have

$$\lim_{\lambda\to 0}\det\begin{pmatrix}\dfrac{G(\lambda u)}{\lambda}\\ DG(\lambda u)(\tau^{(i)})2\leqq i\leqq n\\ D^2G(\lambda u)(v^2)\end{pmatrix}=0, \text{ that is,}$$

$$\det\begin{pmatrix}DG(o)(u)\\ DG(o)(\tau^{(i)})\ 2\leqq i\leqq n\\ D^2G(o)(v^2)\end{pmatrix}=0.$$

We now fix an arbitrary non-zero vector $\eta\in\mathbf{C}^n$ and choose a unitary transformation $\mathbf{V}$ of $\mathbf{C}^n$ such $\mathbf{V}^{-1}\eta$ is in the space spanned by v. Put $\tilde{g}=\tilde{g}\cdot\mathbf{V}$ and $\tilde{G}=G\circ\mathbf{V}$. Since the codition (*) holds for $\tilde{g}$,

$$\det\begin{pmatrix}DG(o)(\mathbf{V}u)\\ DG(o)(\mathbf{V}\tau^{(i)})\ 2\leqq i\leqq n\\ D^2G(o)(\mathbf{V}v)^2)\end{pmatrix}=0$$

Since $\{\mathbf{V}u,\ \mathbf{V}\tau^{(i)},\ 2\leqq i\leqq n\}$ is an orthonormal basis of $\mathbf{C}^n$, it holds that

$$\det\begin{pmatrix}DG(o)(u)\\ DG(o)(\tau^{(i)})\ 2\leqq i\leqq n\\ D^2G(o)(\eta^2)\end{pmatrix}=0.$$

From this equality and from the fact that the rank of $Dg(o)$ is equal to n, it holds that $D^2g(o)(\eta^2)$ is in the spanned by $\{Dg(o)(u),\ Dg(o)(\tau^{(i)}),\ 2\leqq i\leqq n\}$, that is $D^2g(o)(\eta^2)\in Dg(o)(\mathbf{C}^n)$ for any $\eta\in\mathbf{C}^n$. Take a positive number $r<1$ such that for $z\in B_n$ with $|z|<r$, the rank of $Dg(z)$ is equal to n and fix any of this z. Let $\tilde{\psi}\in\mathrm{Aut}\,(B_k)$, $\tilde{\phi}\in\mathrm{Aut}\,(B_n)$ be involutions such that $\tilde{\phi}(o)=z$ and that $\tilde{\psi}\circ g\circ\tilde{\phi}(o)=o$. Put $h=\tilde{\psi}\circ g\circ\tilde{\phi}$. Since the condition (*) holds for the mapping h, by the same method, it holds that $D^2h(o)(\eta^2)$ is in the space $Dh(o)(\mathbf{C}^n)$ for any $\eta\in\mathbf{C}^n$. Since $\tilde{\psi}$ and $\tilde{\phi}$ are involutions, by Remark 5 of section 1, it holds that $D\tilde{\psi}(g(z))(D^2g(z)((D\tilde{\phi}(o)(\eta)^2))$ is in the space spanned by $\{D\tilde{\psi}(g(z))(Dg(z)(D\tilde{\phi}(o)(u)),\ D\tilde{\psi}(g(z))(Dg(z)(D\tilde{\phi}(o)(\tau^{(i)}))),\ 2\leqq i\leqq n\}$.
Since $D\tilde{\psi}(g(z))$ and $D\tilde{\phi}(o)$ are non-singular and since $\{D\phi(o)(u),\ D\phi(o)(\tau^{(i)})\ 2\leqq i\leqq n\}$ are lineary independent, $D^2g(z)(\eta^2)$ is in the space spanned by $\{Dg(z)(u),\ Dg(z)(\tau^{(i)})\ 2\leqq i\leqq n\}$ for any $\eta\in\mathbf{C}^n$. Then

$$\det\begin{pmatrix}DG(z)(u)\\ DG(z)(\tau^{(i)})\ 2\leqq i\leqq n\\ D^2G(z)(\eta^2)\end{pmatrix}=0$$

for any $\eta\in\mathbf{C}^n$. By considering the vector $\xi+\eta$ instead of η, we have

$$\det\begin{pmatrix}DG(z)(u)\\ DG(z)(\tau^{(i)})\ 2\leqq i\leqq n\\ D^2G)z)(\xi,\ \eta)\end{pmatrix}=0$$

for any ξ and η. By the same manner,

$$\det\begin{pmatrix} D^2G(z)(\tau,\ \omega) \\ DG(z)(\tau^{(i)})\ 2\leqq i\leqq n \\ D^2G(z)(\eta^2)\end{pmatrix}=0$$

for any τ, ω, and η. This equality holds for all z with $|z|<r$. For a fixed non-zero vector ξ, let

$$h(\lambda)=\det\begin{pmatrix} DG(\lambda\xi)(u) \\ DG(\lambda\xi)(\tau^{(i)})\ 2\leqq i\leqq n \\ D^2G(\lambda\xi)(\xi^2)\end{pmatrix}.$$

Then $h(\lambda)$ is holomorphic in $|\lambda|<\frac{1}{|\xi|}$ and vanishes in $|\lambda|<\frac{r}{|\xi|}$, so that vanishes in $|\lambda|<\frac{1}{|\xi|}$. By differentiation of $h(\lambda)$,

$$\det\begin{pmatrix} DG(\lambda\xi)(u) \\ DG(\lambda\xi)(\tau^{(i0}\ 2\leqq i\leqq n \\ D^3G)\lambda\xi)(\xi^3)\end{pmatrix}=0, \text{ so that}$$

$$\det\begin{pmatrix} DG(o)(u) \\ DG(o)(\tau^{(i)}\ 2\leqq i\leqq n \\ D^3G(o)(\xi^3)\end{pmatrix}=0.$$

From this, it holds that $D^3g(o)(\xi^3)\in Dg(o)(\mathbf{C}^n)$. By the same way it holds that

$$D^{\ell}g(o)(\xi^{\ell})\in Dg(o)(\mathbf{C}^n)$$

for any $\xi\in\mathbf{C}^n$ and for any positive integer ℓ. Therefore,

$$g(\xi)=\sum\frac{D^{\ell}g(o)(\xi^{\ell})}{\ell!}\in Dg(o)(\mathbf{C}^n),$$

so that $g(B_n)\subset Dg(o)(\mathbf{C}^n)$, a linear subspace of $\mathbf{C}^k$ of dimension n.

Let $\mathbf{U}$ be a unitary transformation of $\mathbf{C}^k$ such that $\mathbf{U}(Dg(o))(\mathbf{C}^n))=\{w_1, \cdots, w_k)\in\mathbf{C}^k$; $w_{n+1}=0, \cdots, w_k=0\}$. Put $\mathbf{U}\circ g(z)=(\tilde{g}_1(z), \cdots, \tilde{g}_k(z))$, then $\tilde{g}_{n+1}(z)=0, \cdots, \tilde{g}_k(z)=0$ for any $z\in B_n$. Hence $\mathbf{V}(z)=(\tilde{g}_1(z), \cdots, \tilde{g}_n(z))$ is a biholomorphic mapping from B_n onto B_n sending the origin to the origin, so that $\mathbf{V}(z)$ is a unitary transformation of $\mathbf{C}^n$. Put $\chi(z)=\mathbf{U}\circ g\circ\mathbf{V}^{-1}(z)=\mathbf{U}\circ\psi\circ f\circ\mathbf{V}^{-1}(z)$, then $\chi(z)=(z, o')$. Let $\mathbf{V}(a)=b$, $\mathbf{U}\psi(o)=\tilde{a}$, then since $f(o)=o$, $\chi(b)=\mathbf{U}\psi(o)=\tilde{a}$, so that $\tilde{a}=(b, o')$. Then it is easily seen that there exists unitary transformations $\tilde{\mathbf{V}}$ of $\mathbf{C}^n$, $\tilde{\mathbf{U}}$ of $\mathbf{C}^k$ and involutions $\tilde{\phi}$ of B_n and $\tilde{\psi}$ of B_k with $\tilde{\phi}(b)=o$, $\tilde{\psi}(\tilde{a})=o$ such that $f=\tilde{\mathbf{U}}\circ\tilde{\psi}\circ\chi\circ\tilde{\phi}\circ\tilde{\mathbf{V}}$ (see Rudin [2] page 28). By Remark 1 of section 2, $\tilde{\psi}\circ\chi\circ\tilde{\phi}\circ\tilde{\mathbf{V}}(z)=(\tilde{\mathbf{V}}(z), o')$, so that $f(z)=\tilde{\mathbf{U}}((\tilde{\mathbf{V}}(z), o'))$ and this is a linear isometry mapping.

By an easy calculation, it follows that

Lemma. $D^2(\psi\circ f\circ\phi)(z)=D^2\psi(f\circ\phi(z))((Df(\phi(z))(D\phi(z)(v)))^2)$
$+D\psi(f\circ\phi(z))(D^2f(\phi(z))((D\phi(z)(v))^2))$
$+D\psi(f\circ\phi(z))(Df(\phi(z))(D^2\phi(z)(v^2)))$.

It is clear from this lemma that if f is linear and isometry, then the condition (*) mentioned previously is satisfied. Consequently, we obtained the following

Theorem. Let f be a proper holomorphic mapping from B_n to B_k $(k>n)$ which is of class C^2 on $\bar{\mathrm{B}}_n$ and $f(0)=0$. Then f is linear isometry if and only if there exists a non-zero vector v of $\mathbf{C}^n$ such that for any choice of an automorpism ϕ of B_n and a ψ of B_k with $\psi\circ f\circ\phi(0)=0$, the following condition (*) is satisfied:

(*) $\mathrm{D}^2(\psi\circ f\circ\phi)(u)(v^2)$ *is in the space spanned by* $\{\psi\circ f\circ\phi(u), \mathrm{D}(\psi\circ f\circ\phi)(u)(\tau^{(2)}), \cdots, \mathrm{D}(\psi\circ f\circ\phi)(u)(\tau^{(n)})\}$.

Reference.

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